

# A Note on the Speed of Perfect State Transfer

Alastair Kay

Department of Mathematics, Royal Holloway University of London, Egham, Surrey, TW20 0EX, UK\*  
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In Phys. Rev. A 74, 030303 (2006), Yung showed that for a one-dimensional spin chain of length  $N$  and maximum coupling strength  $J_{\max}$ , the time  $t_0$  for a quantum state to transfer from one end of the chain to another is bounded by  $J_{\max}t_0 \geq \pi(N-1)/4$  (even  $N$ ) and  $J_{\max}t_0 \geq \pi\sqrt{N^2-1}/4$  (odd  $N$ ). The proof for even  $N$  was elegant, but the proof for odd  $N$  was less so. This note provides a proof for the odd  $N$  case that is simpler, and more in keeping with the proof for the even case.

The benefit of this simplified proof is that it can be used elsewhere. For example, in the study of speed limits for synthesising quantum states using identical Hamiltonian structures [1], where the phase conditions on the eigenvalues are slightly different, these differences are readily incorporated in the optimisation.

A tri-diagonal  $N \times N$  matrix with real diagonal elements  $B_n$  and positive off-diagonal elements  $J_n$ ,

$$h = \sum_{n=1}^N B_n |n\rangle \langle n| + \sum_{n=1}^{N-1} J_n (|n\rangle \langle n+1| + |n+1\rangle \langle n|),$$

is capable of perfect state transfer if and only if  $h$  is symmetric, meaning  $B_n = B_{N+1-n}$  and  $J_n = J_{N-n}$ , and the ordered eigenvalues  $\lambda_n > \lambda_{n+1}$  satisfy  $e^{-i\lambda_n t_0} = (-1)^{n+1} e^{i\phi}$  for some real parameters  $t_0$  (the transfer time) and  $\phi$  [2]. The symmetry imposes that the eigenvectors satisfy  $S|\lambda_n\rangle = (-1)^{n+1}|\lambda_n\rangle$  where

$$S = \sum_{n=1}^N |n\rangle \langle N+1-n|.$$

We assume that  $\sum_n B_n = \sum_n \lambda_n = 0$  because adding  $\mathbb{I}$  to  $h$  only changes the phase  $\phi$ .

*N even:* We repeat Yung's proof [3]. If we calculate  $\text{Tr}(SH)$  for both the matrix directly, and for the eigenvector decomposition, we have

$$2J_{N/2} = \sum_{n=1}^{N/2} \lambda_{2n-1} - \lambda_{2n}.$$

We know that  $\lambda_{2n-1} \geq \lambda_{2n} + \frac{\pi}{t_0}$  in order for it to satisfy the required phase property for the eigenvalues, so

$$2J_{\max} \geq 2J_{N/2} \geq \frac{N}{2} \frac{\pi}{t_0},$$

and we immediately get the claimed relation

$$J_{\max}t_0 \geq \frac{\pi N}{4}.$$

*N odd:* Evaluate  $\text{Tr}(SH^2)$ , which gives

$$4J_{\max}^2 \geq 4J_{(N-1)/2}^2 = \lambda_N^2 + \sum_{n=1}^{(N-1)/2} (\lambda_{2n-1} - \lambda_{2n})(\lambda_{2n-1} + \lambda_{2n}).$$

Again, we use the relation  $\lambda_{2n-1} \geq \lambda_{2n} + \frac{\pi}{t_0}$  to find that

$$4J_{\max}^2 \geq \lambda_N^2 + \frac{\pi}{t_0} \sum_{n=1}^{N-1} \lambda_n = \lambda_N^2 - \frac{\pi}{t_0} \lambda_N.$$

by the assumption  $\sum_n \lambda_n = 0$ . However,  $\lambda_N$  is also constrained; a little thought is sufficient to convince that the largest it can be is  $-(N-1)\pi/(2t_0)$  [4], which yields

$$J_{\max}^2 t_0^2 \geq \frac{\pi^2(N^2-1)}{16},$$

exactly as claimed.

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\* Electronic address: alastair.kay@rhul.ac.uk

[1] A. Kay, arXiv:1609.01398 [quant-ph] (2016), arXiv:1609.01398.

[2] A. Kay, Int. J. Quantum Inform. **8**, 641 (2010).

[3] M. Yung, Phys. Rev. A **74**, 030303 (2006).

[4] Or, substitute  $\lambda_n \geq (N-n)\pi/t_0 + \lambda_N$  in  $\sum_n \lambda_n = 0$ .